DOWNING COLLEGE CAMBRIDGE
Summer School

Mathematics A

Keenan J. A. Down

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Question Sheet 1

Sets

1. (a) Let A and B be finite sets with $|A| = a$, $|B| = b$. Suppose that $|A \cup B| = a + b - n \geqslant 0$. Give an expression for $|A \cap B|$ [Recall that we use $|S|$ to denote the number of elements in a set].
   (b) Give an expression for $|A \cap B \cap C|$ in terms of $|A \cup B \cup C|$, $|A \cup B|$, $|A \cup C|$, $|B \cup C|$, and $|A|$, $|B|$, $|C|$.
   (c) What do you notice about the signs of each of the terms?

2. Consider the family of sets
$$P_2 = \{2\}$$
$$P_n = P_{n-1} \cup \{\, p \in \mathbb{N} : 2 \leqslant p \leqslant n, \text{and for all } q \in P_{n-1},\, q \text{ does not divide } p\,\}.$$
Write out in full, using roster notation $\{\}$, the elements contained in P_{25}.

3. An infinite set S is called **countably infinite** if its elements can be written out in a list. For example, $\mathbb{N}$ is countably infinite as $\mathbb{N}$ can be written as a list $\{1, 2, 3, 4, \ldots\}$.
   (a) Construct a list to show that $\mathbb{Z}$ is countably infinite.
   (b) Construct a list to show that $\mathbb{Q}$, the rational numbers, is countably infinite.
   (c) Do you think that $\mathbb{R}$ is countably infinite?

4. One problem you might not have noticed is that our naive approach to defining sets sometimes allows us to create sets which might be rather contradictory. Consider this:

   Let S be the set of all sets which do not contain themselves.

   At first this might seem like a valid set. However, one question suddenly becomes impossible to answer:

   Does S contain S?

   This problem, called **Russel's paradox**, has been resolved by modern approaches to set theory, but the first attempt at set theory (called **Naive set theory**) did raise issues such as this. Prove that this statement is contradictory.

5. Let A and B be two finite sets. The **Cartesian product** of A and B, written $A \times B$, is the set of ordered pairs from A and B. That is,
$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

(a) Give an expression for $|A \times B|$.

(b) Let $f : A \times A \to A$. Is it possible that f is injective?

Functions and Relations

6. For each of the following functions, determine whether or not they are (a) Injective (b) Surjective.

   (a) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$.

   (b) $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = x^2$.

   (c) $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 3x$.

   (d) $f : \mathbb{Z} \times \mathbb{N}^+ \to \mathbb{Q}$ where $f(a, b) = a/b$. [Note: $\mathbb{Z} \times \mathbb{N}^+$ means we take one element from both of these sets. Recall also that $0 \notin \mathbb{N}^+$.]

7. For each of the following relations, determine whether or not they are (a) Symmetric, (b) Reflexive, (c) Transitive:

   (a) Let $a, b \in \mathbb{R}$, then $a \sim b$ if $a \leqslant b$.

   (b) Let $a, b \in \mathbb{R}$, then $a \sim b$ if $a < b$.

   (c) Let x and y be strings (i.e. collections of characters), with $x \sim y$ if x appears at or before y in the dictionary.

   (d) Let x and y be strings, with $x \sim y$ if x can be changed into y by adding, changing, or deleting exactly one character.

8. Let $f : \mathbb{Z} \to \mathbb{Z}$ where

$$f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ 3x + 1 & x \text{ is odd.} \end{cases}$$

   (a) Compute $f^9(64)$. What do you notice?

   (b) Compute $f^7(3)$.

   (c) Compute $f^n(7)$, for $n = 10$ and explain how the sequence will continue as n increases.

   You might have realised that every number we tested so far eventually returns to 1. Do you think this is true for all $n \in \mathbb{N}$? This is actually an important unsolved problem in mathematics called the **Collatz conjecture**. The conjecture has been shown to be true for the first $\approx 2.95 \times 10^{20}$ numbers, but we still haven't managed to prove the result!

9. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = ax^2 + bx + c$ is injective if and only if $a = 0$ and $b \neq 0$.

10. (**Hard**) Recall that, given an equivalence relation $\sim$ on a set S, an **equivalence class** is a maximal subset $C \subseteq S$ such that all of the elements in C are equivalent, i.e. $a \sim b$ for all $a, b \in C$.

    Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Let $\sim$ be a relation defined by

$$a \sim b \text{ if } f(a) = f(b).$$

    Given some element $s \in S$, we will denote the equivalence class of s by $\bar{s}$. Let E be the set of equivalence classes on $\mathbb{R}$ defined by $\sim$.

    Now define a new function $\bar{f} : E \to \mathbb{R}$ such that $\bar{f}(\bar{s}) = f(s)$ for each equivalence class.

    (a) Show that $\bar{f}$ is well defined. I.e. for two elements $a, b \in S$ with $a \sim b$, that $\bar{f}(\bar{a}) = \bar{f}(\bar{b})$.

    (b) Prove that the new function $\bar{f}$ is injective on E.

Question Sheet 2

Logic and Statements

1. For each the following pairs of statements, determine whether the first is (i) Sufficient (ii) Necessary for the second.

(a)	Harry is a Burmese cat.	Harry has fur.
(b)	Ethanol is a gas at 100° C.	Ethanol has a boiling point of 78° C.
(c)	x is a multiple of 6.	x is a multiple of 2 and a multiple of 3.
(d)	John thinks Jane is cool.	Jane thinks John is cool.
(e)	I am a vegetarian.	I don't consume meat.

2. Let $f(x) = x^2 + bx + c$ for some $b, c \in \mathbb{R}$. Prove that the following two statements are equivalent:

   - For all $x \in \mathbb{R}$, $f(x) > 0$.
   - $b^2 - 4c < 0$.

3. Suppose we have 3 statements A, B and C which we want to prove are equivalent. We can only prove one implication $\implies$ at a time. How can we show the three statements are equivalent with the lowest number of implications? What if we had n equivalent statements?

Negations

4. Construct a truth table on A and B for the following statement:

$$\neg(A \wedge B) \iff \neg A \vee \neg B$$

What do you notice? Here is a similar statement:

$$\neg(A \vee B) \iff \neg A \wedge \neg B.$$

These two statements are called **De Morgan's Laws**. Use De Morgan's laws to negate the statements

Harriet is a banker and Janet is her boss.

Either you're a good liar, or you're incredibly persuasive.

5. Negate the following statements:

   (a) $\forall\, x > 0, \exists\, y > 0\, [x - y = f(x)]$.

   (b) $\exists\, n \in \mathbb{N}\, [\,\forall\, m \geqslant n\, [a_n + a_{n-1} \leqslant b(m)]]$

   (c) $\forall x, y \in \mathbb{R}\, [\, f(x) - f(y) = 0 \implies [f(x)^2 - f(y)^2 - 1 < |x|\,]]$

   (d) $\forall\, a, \exists\, b, \forall\, c, \exists\, d, \forall\, e, \exists\, f\, [\, P(a, b, c, d, e, f)\,]$
   (N.b. $P(a, b, c, d, e, f)$ is some variable statement in terms of a, b, c, d, e, f.)

Many mathematicians will use slightly different notation for statements like these. You might sometimes see 's.t.' (short for 'such that') between the existential quantifier and the universal quantifier. It can also be used in place of the square brackets we've been using.

6. Negate the following statements:

   (a) 'Every dog in Japan is a Shiba Inu'.

   (b) 'After every single magic show I've seen, there was at least one person who thought it was real.'

   (c) 'There is at least one kind of food you don't like.'

   (d) 'Every politician lies, and there is at least one lawyer who doesn't.

Proof

7. (a) Prove the following identity by induction:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (b) Let $a_1 = 1$ and let $a_n = 2a_{n-1} + 1$ for $n \in \mathbb{N}$ with $n \geqslant 2$. Give a short inductive argument to explain why $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Using this, prove using induction that $|a_n| \geqslant 2^{(n-1)}$ for all $n \in \mathbb{N}$.

8. Recall that a **countably infinite** set S is an infinite set whose elements can be written in an infinite list, and recall that the Cartesian product of two sets A and B, denoted $A \times B$ is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

   (a) Let S_1 and S_2 be two countably infinite sets. Prove that the Cartesian product $S_1 \times S_2$ is also countably infinite. [Hint: you might want to use a 2-dimensional argument.]

   (b) Using this fact, prove that any finite Cartesian product $S_1 \times \cdots \times S_n$ is also countably infinite for any $n \in \mathbb{N}$.

9. Recall that given a statement $A \implies B$ that this is logically equivalent to $\neg B \implies \neg A$. This is called proving the **contrapositive**.

   Let A and B be two finite sets with $|A| = |B|$. Use the contrapositive to prove the following statement:

$$f : A \to B \text{ is injective} \implies f \text{ is surjective.}$$

10. (**Hard**). Recall that the **Fibonnaci sequence** is given by $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \in \mathbb{N}$ with $n \geqslant 3$.

   (a) Let $x \in \mathbb{R}^+$ be a positive real number. Find an expression using binomial expansion for

$$(1 + x)^n - (1 - x)^n.$$

   (b) Prove the following formula using induction:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

   This formula is called **Binet's formula**, and you might learn how to derive it in a course on differential equations.

Question Sheet 3

Euclidean Division

1. Using the Division algorithm, compute the highest common factor of 7938 and 4608.

2. You might be surprised to learn that Euclid's algorithm can actually be applied to many different kinds of objects.

   (a) Let $f(x), g(x) \in \mathbb{R}[x]$ (i.e. polynomials in x with real coefficients). Assume further that $\deg(f) \geqslant \deg(g)$. Show that there exists some $q(x)$ and $r(x)$ such that

   $$f(x) = g(x)q(x) + r(x)$$

   and $\deg(r) < \deg(g)$.

   (b) Using the resulting division algorithm for polynomials, compute the highest common factor of

   $$\begin{aligned} f(x) &= x^4 - x^3 + 2x^2 - 9x + 7 \\ g(x) &= 4x^3 - x^2 - 4x + 1. \end{aligned}$$

   The kinds of mathematical spaces where we can perform Euclidean division are called **Euclidean domains**, and they have many useful and important properties.

3. For each of the following pairs of numbers $m, n \in \mathbb{Z}$, find some $a, b \in \mathbb{Z}$ such that

   $$am + bn = \mathrm{hcf}(m, n).$$

   (a) 5 and 21.

   (b) 25 and 55.

   (c) 152 and 106.

   (d) 284 and 53.

   Now suppose that we have some number m and a prime number p where p does not divide m. Explain why we can always find some expression

   $$am + bp = 1$$

   where $a, b \in \mathbb{Z}$.

Modular Arithmetic

4. Using the results of question 3, show that in arithmetic $\bmod p$, any number $x \in \{1, \ldots, p-1\}$ has a multiplicative inverse. That is to say, show that there exists some $y \in \{0, \ldots, p-1\}$ such that

   $$xy = 1 \bmod p.$$

   Notice that 0 is the only number which does not have a multiplicative inverse (much like how we cannot divide by 0 in $\mathbb{R}$). If every number has a multiplicative inverse, this is one way of expressing that the integers $\bmod p$ are a kind of object called a **field**, where every number can be added, subtracted, multiplied and divided (except 0).

   Explain for which values of n the set $\mathbb{Z}_n$ is a field.

5. For this question, we're going to look at polynomials with coefficients in $\mathbb{Z}_3$.

(a) Consider the following polynomial $f \in \mathbb{Z}[x]$:

$$f(x) = 7x^2 + 11x + 2$$

By considering this polynomial $\bmod 3$, show that for all $x \in \mathbb{Z}$, $f(x)$ is not divisible by 3. Hence explain why the equation $f(x) = 0$ has no solutions in $\mathbb{Z}$.

(b) Consider the polynomial

$$g(x) = x^2 + 1.$$

Prove that given some $x \in \mathbb{Z}$, we must always have that

$$g(f(x)) = 3n + 2 \quad \text{for some } n \in \mathbb{Z}.$$

6. (a) Is $30^{30} + 1$ divisible by 12?
   (b) What is $31^{31} \bmod 12$?
   (c) What is the final digit of 1089^{1089}?
   (d) Compute $x^6 \bmod 7$ for $x \in \{0, 1, 2, 3, 4, 5, 6\}$. What do you notice? After tomorrow's lecture you might be able to prove your conjecture.

7. We saw in lecture 2 that $\sqrt{2}$ is irrational. Use a proof by contradiction and arithmetic $\bmod 6$ to prove that the fraction

$$\frac{\ln(3)}{\ln(6)}$$

must be irrational. How does your argument fail for $\ln(2)/\ln(4)$?

8. Euclid proved that there must be infinitely many prime numbers. Prove that there must be infinitely many prime numbers.

Rings and Integral Domains

9. (**Very Hard**). We saw earlier that there are some alternative objects on which we can apply the Euclidean algorithm. Suppose we have some system of arithmetic in which we can add, subtract and multiply (not necessarily divide). The technical name for an object like this is a **ring**. You can read more about rings online and see their complete definition.

Given some **ring** R, we say that R is an **integral domain** if multiplication in $\mathbb{R}$ is commutative (i.e. $ab = ba$ for $a, b \in R$), and given two nonzero elements $x, y \in R$ we must have $xy \neq 0$ (this is sometimes phrased as R not having any **zero-divisors**).

An **integral domain** is called a **Euclidean domain** if there is some function $f : R \to \mathbb{N}$ such that given any $a, b \in R$ with $b \neq 0$, then there exists some $q, r \in R$ with

$$a = bq + r$$

with either $f(r) < f(b)$ or $f(r) = 0$.

   (a) Can you think of an example of a ring which is **not** an integral domain?
   (b) What was $f(x)$ in the polynomial ring in question 2?
   (c) Using the well-ordering principle, prove the existence step of Euclid's division lemma for general Euclidean domains.
   (d) Is the remainder r unique in general Euclidean domains?

10. (**Hard**). In the lecture we mentioned the **Fundamental Theorem of Arithmetic**. This result demonstrated that every natural number could be factorised *uniquely* into a product of prime numbers.

In **algebraic number theory**, we often work with more general domains where, like in the integers, factorisation is possible. As it turns out, some of these domains (called **Unique Factorisation Domains**, UFDs) have unique factorisation, but in general this isn't the case.

(a) Consider the **Gaussian integers** $\mathbb{Z}[i]$. That is

$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

Prove that there exists some function $f(z) : \mathbb{Z}[i] \to \mathbb{R}^+ \cup \{0\}$ which satisfies the requirements of a Euclidean function.

(b) It is known that all Euclidean domains are unique factorisation domains.

i. Consider the integral domain

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

Is factorisation unique in this domain?

ii. Using your answer to part (i), ascertain whether or not the integral domain $\mathbb{Z}[\sqrt{-5}]$ is a Euclidean domain.

Question Sheet 4

Groups

1. Prove that the following sets and their binary operations are **groups**.
   (a) $(\mathbb{Z}, +)$.
   (b) $(\mathbb{Q}^*, \times)$.
   (c) $(R, +)$ where $R = \{f : \mathbb{R} \to \mathbb{R} \mid f(1) = 0\}$.
   (d) $(S, \times)$ where $S = \{z \in \mathbb{C} : |z| = 1\}$. If you haven't met the complex numbers yet, here is one definition:
   $$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}.$$
   and we define
   $$|a + bi| = \sqrt{a^2 + b^2}.$$
   (e) The set of all possible rotations $R = \{0 \leqslant \rho < 2\pi\}$ of the unit circle under composition.
2. In this question, we'll talk about constructing larger groups out of smaller groups.
   (a) Show that $(\mathbb{R}^2, +)$ is a group.
   (b) Suppose that G_1 and G_2 are two groups with operations $\times_1$ and $\times_2$ respectively. Show that
   $$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$$
   is a group under the operation
   $$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \times_1 g_2, h_1 \times_2 h_2).$$
3. Consider each of the following examples of sets and binary operations. Are they groups? If they are groups, are they abelian?
   (a) The set of rotational symmetries of any 2d-polygon P, under composition.
   (b) The set of all functions $f : \mathbb{R} \to \mathbb{R}$ under multiplication.
   (c) The set $\mathbb{Z}$ but with $a * b = a + b + 1$.
   (d) The set of all strings under concatenation.
4. Let D be the group of symmetries of the triangle (a Dihedral group).
   (a) What is $|D|$?
   (b) Draw a complete multiplication table for $|D|$.
5. Consider the following set:
   $$G = \{1, x, y, z\}$$
   with $x^2 = y^2 = z^2 = 1$.
   (a) G is a group under multiplication. Using this fact, deduce what element is given by the products xy, yz and zx.
   (b) Using your answer, draw a multiplication table for G.
   (c) Determine and prove whether or not G is an abelian group.
6. Recall the group $(\mathbb{Z}_7, +)$ of integers modulo 7 with addition. We're going to consider the same set $\mathbb{Z}_7$, but now with multiplication.
   (a) Does 3 have a multiplicative inverse in $\mathbb{Z}_7$? Show that there is another number x such that $3x = 1$ in $\mathbb{Z}_7$.
   (b) We've just shown that 3 has a multiplicative inverse. What is the multiplicative group generated by 3? Write out $\langle 3 \rangle$ explicitly. What do you notice?

7. In lecture 3 we saw Bézout's identity. Furthermore, in the third problem sheet we saw that we could use Bézouts identity to retrieve a useful result. In particular, we saw that for any prime p and for any $x \in \{1, 2, \ldots, p-1\}$, there existed some x^{-1} such that

$$x \cdot x^{-1} \equiv x^{-1}x \equiv 1 \bmod p.$$

Use this fact to prove that the set

$$\mathbb{Z}_p^* = \{1, \ldots, p-1\}$$

is a group under multiplication.

8. (**Hard**). Suppose we have some multiplicative group G such that for every element $g \in G$, $g^2 = 1$. Prove that G must be abelian.

Subgroups, Cosets and Lagrange's Theorem

9. Consider the group $G = (\mathbb{Z}, +)$ of integers under addition and the subgroup $H = (n\mathbb{Z}, +)$ for some fixed $n \in \mathbb{N}$, where

$$n\mathbb{Z} = \{nz : z \in \mathbb{Z}\}.$$

(a) Prove that H is a subgroup of G.

(b) What is the index $[G : H]$ of H in G?

(c) Suppose now we have two subgroups $H_n = (n\mathbb{Z}, +)$ and $H_m = (m\mathbb{Z}, +)$. Consider the set

$$H_n + H_m = \{nx + my : x, y \in \mathbb{Z}\}.$$

This is a subgroup of $\mathbb{Z}$. Show that there exists some $k \in \mathbb{N}$ such that

$$H_n + H_m = H_k.$$

(d) Give a brief argument to explain why **all subgroups** of $\mathbb{Z}$ are of the form $a\mathbb{Z}$ for some $a \in \mathbb{N}$.

10. (**Hard**). Recall that the multiplicative subgroup generated by an element $g \in G$ is given by

$$\langle g \rangle = \{\ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots\} \leqslant G.$$

In the previous question we saw that $(\mathbb{Z}_p^*, \times)$ is a multiplicative group for any prime p. By considering each element $1 \leqslant x \leqslant p-1$ with $x \in \mathbb{Z}_p^*$, and the subgroup it generates $\langle x \rangle$, use **Lagrange's theorem** to prove the following result:

Theorem 1 (Fermat's Little Theorem)**.**
For all $x \geqslant 1 \in \mathbb{N}$ and any prime p,

$$x^{p-1} \equiv 1 \bmod p.$$

Question Sheet 5

Sequences and Series

1. For each of the following sequences, do they **converge** or **diverge**? If they converge, what are they converging to? [Note that in this question we take all sequences to start at $n = 1$.]

   (a) $a_n = \frac{n+1}{n^2+1}$.

   (b) $a_n = \frac{3n^2+2n}{n^2+1}$.

   (c) $a_n = \sin(1/n) + \cos(1/n)$.

   (d) $a_n = \sin(\sin(\sin(\sin(1/n))))$.

   (e) $a_n = \ln(n)$.

   (f) $a_n = \ln(\ln(n))$.

2. For each of the following series, do they **converge** or **diverge** If they converge, what are they converging to?

   (a) $S_n = \sum_{k=1}^{n} 2^{-k}$.

   (b) $S_n = \sum_{k=1}^{n} (-1)^k$.

   (c) $S_n = \sum_{k=1}^{n} \frac{1}{\ln(k+1)}$.

   (d) $S_n = \sum_{k=1}^{n} \sin(k)$.

   (e) $S_n = \sum_{k=1}^{n} \sin(\pi k)$.

3. Use the fact that for small x, $\sin(x) \approx x$ to explain why the series

$$S_n = \sum_{k=1}^{n} \sin\left(\frac{1}{k}\right)$$

   does not converge.

4. The **triangle inequality** states that for any two numbers $x, y \in \mathbb{R}$ we have

$$|x + y| \leqslant |x| + |y|.$$

   We say that a series

$$S_n = \sum_{k=1}^{n} f(k)$$

   is **absolutely convergent** if the series

$$S'_n = \sum_{k=1}^{n} |f(k)|$$

   converges. Use the triangle inequality to demonstrate that an *absolutely convergent* series is, in fact, *convergent*.

5. Suppose that we have some series $S_n = \sum_{k=1}^{n} (-1)^n f(k)$ where $f(k) \geqslant 0$ for all k.

   (a) Show that if $f(k+1) < f(k)$ for all $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} f(k) = 0 \implies S_n \text{ is convergent.}$$

   (b) Hence show that the series

$$S_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k}$$

   is convergent, but not absolutely convergent.

Limits and Continuity

6. Evaluate if each of these limits exists. If it does, what is the limit?

   (a) $\lim_{x \to 10} \frac{1}{x^2 - 100}$.

   (b) $\lim_{x \to \pi} \ln(x)$.

   (c) $\lim_{x \to \frac{\pi}{2}} \frac{\sin(x)}{\cos(x)}$.

   (d) $\lim_{x \to 0} \frac{\sin(x)}{x}$.

   (e) $\lim_{x \to \infty} \frac{x^2+1}{2x^2-1} + \frac{x^4}{x^5-x^3-x}$.

7. Use the limit version of continuity to ascertain whether or not the following functions are continuous:

   (a) $f(x) = x^3 - 2x$.

   (b) $f(x) = \frac{1}{|x|+1}$.

   (c)
   $$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

   (d) $f(x) = e^x$.

   (e) $f(x) = \frac{1}{x}$ on $\mathbb{R}^*$.

8. In this question, we'll explore how to re-derive the chain rule from first principles.

   (a) Using the formal definition of limit, compute the derivative of x^3. What do you notice about the terms which went to zero in the limit?

   (b) Using the binomial expansion formula
   $$(x+y)^n = \sum_{k=0}^{n} \binom{x}{y} x^{n-k} y^k,$$
   derive an expression for $\frac{\mathrm{d}}{\mathrm{d}x} x^n$.

9. Using the limit definition for continuity, prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two continuous functions, then their composition
   $$f \circ g = f(g(x))$$
   is also continuous.

10. (**Hard**). Using the formal definition of the derivative, re-derive the quotient rule
    $$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Question Sheet 6

Linear Maps and Vector Spaces

1. Explain whether or not each of the following functions is **linear**:
   (a) $f(x) : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x$.
   (b) $f(x) : \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (2x + 1, 5x - 1)$.
   (c) $f(x) : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x, y) = (2xy + 2x, x + y)$.
   (d) Let $f(x) = 3x + 2$, and let $g(x) = 2x - 1$. Is $f(x)g(x)$ linear?
   (e) How about $f(g(x))$?
2. Prove that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.
3. Consider the following. Which are vector spaces and which are not?
   (a) $\mathbb{C}^2$ over $\mathbb{C}$.
   (b) $\mathbb{Q}$ over $\mathbb{R}$.
   (c) $\mathbb{C}$ over $\mathbb{R}$.
   (d) $\{a + b\sqrt{2} : a, b, \in \mathbb{Q}\}$ over $\mathbb{Q}$.
   (e) $\mathbb{Z}$ over $\mathbb{R}$.
   (f) $\mathbb{Z}^2$ over $\mathbb{Z}$.
4. Consider the following matrices $\boldsymbol{M}$. Compute the set of vectors $\boldsymbol{v}$ for which $\boldsymbol{M}\boldsymbol{v} = \boldsymbol{0}$.
   (a)
   $$\boldsymbol{M} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$$
   (b)
   $$\boldsymbol{M} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$
   (c)
   $$\boldsymbol{M} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
   What do you notice about the dimensions of the spaces you found?

Matrices for Linear Systems

5. Consider the following matrix:
   $$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   (a) Compute $\boldsymbol{X}^n$ for $n = 2, 3, 4$.
   (b) Hence compute $\boldsymbol{X}^3 - 2\boldsymbol{X}$.
   (c) Hence compute $\boldsymbol{X}^4 - 4\boldsymbol{X}^3 + 6\boldsymbol{X}^2 - 4\boldsymbol{X} + \boldsymbol{I}$. What do you notice? This is called the **characteristic polynomial** of the matrix.
6. Represent the following simultaneous equations using a matrix. Hence construct inverse matrices and solve the simultaneous equations.

(a)

$$14 = 2x + 3y$$
$$20 = 3x + 4y$$

(b)

$$-2 = 3x - 4y$$
$$-10 = -7x + 2y$$

(c)

$$\pi = 2x - 5y$$
$$e = 3x + 12y$$

7. Consider the following matrix:

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) By considering what happens to the point $(1, 0, 0, 0, 0)$, evaluate the determinant of this matrix.
   (b) Is this matrix invertible?
   (c) Compute $\boldsymbol{M}^n$ for $n = 2, 3, 4, 5$. What do you notice? Matrices with this property are called **nilpotent** matrices.

Matrix Algebra

8. Consider the set
$$S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R}, a, c \neq 0 \right\}$$
Show that this is a group under matrix multiplication.

9. (a) Consider the following matrix with entries in $\mathbb{Z}$.
$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
What conditions can we apply to a and b so we can be certain that there exists some $c, d \in \mathbb{Z}$ with $\det(\boldsymbol{A}) = 2$.

   (b) Consider the following matrix with entries in $\mathbb{Z}$.
$$\boldsymbol{B} = \begin{bmatrix} a & p \\ b & c \end{bmatrix}$$
where $a, b, c, p \in \mathbb{Z}$ and c is a prime number. What requirements are there on c such that there exists some $a, b \in \mathbb{Z}$ where $\boldsymbol{B} \in \mathrm{SL}_2(\mathbb{Z})$?

10. (**Hard**). For each of the elements of the dihedral group D_8 (the symmetries of the square), find a 2×2 matrix which represents it.

Question Sheet 7

Iterative Methods

1. For each of the following equations, create an iterative function $g(x_n) = x_{n+1}$. Determine if your iterative function converges by iterating to find a solution. Unless otherwise specified, take $x_1 = 1$.

   (a) $x^2 - 3x - 3 = 0$.

   (b) $x^3 + 10x - 2 = x^4$.

   (c) $x^2 = \sin(x) + 1$.

   (d) $\sin(x) = \cos^2(x) + \frac{x^2}{10}$ starting with $x_1 = -1$.

   (e) $x^5 - x^4 + \pi x^3 + x^2 - 2x + 3 = 0$.

2. Using the **Newton-Raphson method**, create iterative formulae to iteratively find a solution to the following equations:

   (a) $x^2 - 2x - 1 = 0$ starting with $x_1 = 2$.

   (b) $x^5 - x^4 + \pi x^3 + x^2 - 2x + 3 = 0$ starting with $x_1 = -2$.

   (c) $\sin^2(x) - 2x + 1 = 0$ starting with $x_1 = 1$.

   (d) $e^x - 5^x + 4 = 0$ starting with $x_1 = 1$.

   (e) $x - +1 = 0$ starting with $x_1 = -1/2$.

3. Draw a spider diagram to show the path of convergence in the Newton-Raphson method for

   $$f(x) = \sin x = 0$$

   starting at $x_1 = 2$. To what number is this sequence converging?

Completeness

4. Using the notion of **completeness** that a set is **complete** if it contains all of its limit points, explain whether or not each of the following sets is **complete**.

   (a) The complex numbers $\mathbb{C}$.

   (b) Pairs of real numbers $\mathbb{R}^2$.

   (c) The rational numbers $\mathbb{Q}$.

   (d) The integers $\mathbb{Z}$.

Supremum and Infimum

5. For each of the following sets or sequences, find (i) its **supremum** (ii) its **infimum**. Again we take $n = 1$ as the first case with $n \in \mathbb{N}$.

   (a) The sequence $a_n = 1/x$.

   (b) The set $S = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, a > b > 0\right\}$.

   (c) The sequence $a_n = \sin(n)$.

   (d) The set $S = \{1 - \varepsilon : 1 > \varepsilon > 0\}$.

   (e) The sequence $a_n = \sin(n) + \cos(n)$. [Hint: recall that adding a sine and a cosine with the same frequency can be written as a single sine or cosine with a phase shift.]

6. Let $A, B \subset \mathbb{R}$ be two bounded sets (i.e. bounded above and below). Let $A + B = \{a + b : a \in A, b \in B\}$. Show that

   $$\sup A + \sup B = \sup(A + B).$$

The Intermediate Value Theorem

7. For each of the following continuous functions $f(x) : \mathbb{R} \to \mathbb{R}$, locate a solution $f(x) = 0$ by bisecting the interval 4 times.

   (a) $f(x) = x^2 - 2$. Start with the interval $[1, 2]$.

   (b) $f(x) = x^3 - 2x - 2$. Start with the interval $[0, 4]$.

   (c) $f(x) = \sin x + x^3 - 4x - 2$. Start with the interval $[2, 3]$.

8. Let
$$f(x) = \sin x + x \quad \text{and} \quad g(x) = 5 - x^2.$$
   Use the intermediate value theorem to demonstrate:

   (a) There exists a point $c \in [0, 4]$ such that $f(x) = g(x)$.

   (b) There exists a point $c' \in [-4, 0]$ such that $f'(x) = g'(x)$.

Riemann Integration

9. Approximate each of the following integrals using a Riemann sum with (i) 3 strips (ii) 6 strips.

   (a)
$$\int_0^3 x^2 - x + 1 \, \mathrm{d}x.$$

   (b)
$$\int_0^6 e^{\sin x} \, \mathrm{d}x.$$

   (c)
$$\int_0^{2\pi} \cos(x) \, \mathrm{d}x.$$

   (d)
$$\int_0^{2\pi} \sin(\sin(\sin(x))) \, \mathrm{d}x.$$

The Fundamental Theorem of Calculus

10. Using the **fundamental theorem of calculus**, answer the following questions.

    (a) Let
$$f(x) = \int_2^x \sin(t) + 3t^2 - 5 \, \mathrm{d}t.$$
    what is $f'(\pi)$?

    (b) Compute the derivative of $f(x) = \sin x + 5x^2 + 2$. Hence evaluate
$$\int_0^{\pi} \cos(x) + 10x \, \mathrm{d}x.$$

Question Sheet 8

Differentiation

1. In this question we're going to derive the derivative of $\sin x$ from first principles.

   (a) By drawing diagrams, convince yourself that
   $$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{1-\cos x}{x} = 0.$$

   (b) Using the addition formula for $\sin(a+b)$ and the laws for manipulating limits, use the definition of the derivative to derive $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$.

2. In the lecture we saw a counterintuitive function, the Weierstrass function, which is continuous everywhere but differentiable nowhere. In this question we're going to explore another counterintuitive function, called the **cantor function** or the **Devil's staircase**. We will consider this function $f : [0,1] \to [0,1]$.

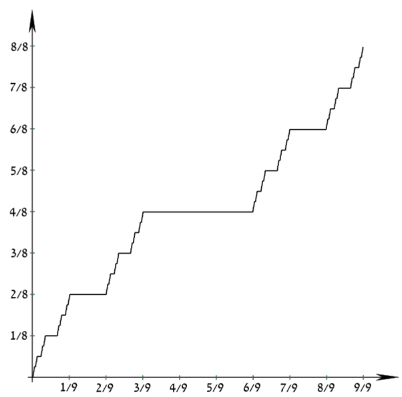

Figure 1: Cantor's function on $[0,1]$
Credit: users Theon and Amirki, wikimedia (CC license).

   To construct this function $f(x)$, we express x in base 3. After the first 1 in x we swap all remaining digits by 0. Then we replace any 2s with 1s. We then interpret the result as a binary number.

   (a) What is $f(0.8)$?

   (b) What do you notice about the derivative of f?

   (c) Let S be the set of points $x \in [0,1]$ where f is not differentiable at x. Explain how to demonstrate that S is a countably infinite set.

Topology

3. In the last lecture we saw that coffee cups and doughnuts can be continuously deformed into each other. For each of these pairs of objects, decide if they are topologically equivalent

   (a) A coffee cup and a ring.

   (b) A coat hanger and a hula-hoop.

   (c) The letter B and the letter Q.

   (d) The letter A and the letter P.

Number Theory

4. In the course we saw a trick to show that the square root of 2 is irrational. We're going to broaden this argument.

   (a) Using a similar strategy to in the lectures, prove that the square root of 3 is irrational.

   (b) Let $n \in \mathbb{N}$ be any number which is not a cube. Show that $\sqrt[3]{n}$ is irrational.

   (c) Let $n \in \mathbb{N}$ be any number with $n \neq m^k$ for $m \in \mathbb{N}, k \in \mathbb{N}, k \geqslant 2$. Show that $\sqrt[k]{n}$ is irrational.

5. Earlier in the lectures we used modular arithmetic to compute the last digits of large numbers.

   (a) As a warm up, compute
   $$64^{37} \bmod 7.$$

   (b) Compute the last digit of 2027^{2027}.

   (c) You might have noticed a pattern above. Using this, compute the last digit of $2027^{2027^{2027}}$.

   (d) How about $2027^{2027^{2027^{2027}}}$?

Series

6. (a) Consider the infinite series
   $$S_1 = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots.$$
   This series diverges, so normally we don't allow ourselves to assign this series with a value. **However**, under certain circumstances we can try and assign these sequences with values. Use an argument by manipulating S_1 to find a value for the series. This series is called **Grandi's series**.

   (b) Using a similar trick, assign a value to the divergent series
   $$S_2 = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots.$$

   (c) Using the two results above, find a trick to assign a value to the divergent series
   $$S_3 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \ldots.$$
   This process is sometimes called **Ramanujan summation**, and it also appears under the guise of **analytic continuation** for the Riemann zeta function. Remarkably, the expression for S_3 appears sometimes in string theory, where this result is actually used.

Linear Algebra

7. For this question we're going to treat some complex numbers like a vector space over $\mathbb{Q}$. Consider the set
   $$\mathbb{Q}[i] = \{a + bi : a, b \in \mathbb{Q}\}.$$

   (a) The space $\mathbb{Q}[i]$ is a vector space. It is 2-dimensional, meaning that it needs two numbers to specify a point. For that reason we'll take a as our first coordinate and b as our second coordinate. Using this coordinate system, express (i) 1 (ii) i (iii) $2+3i$ as column vectors in this vector space.

   (b) Suppose we take a complex number $a + bi$ and multiply by i. This is a **linear map** $f : \mathbb{Q}[i] \to \mathbb{Q}[i]$. Using the coordinate system above, express this linear map as a matrix.

   (c) What is the determinant of this matrix?

   (d) Now write the matrix representing multiplication by 2. What is the determinant of this matrix?

8. Let $\boldsymbol{A}$ be the matrix given by

$$\boldsymbol{A} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}.$$

   (a) Use induction to show that

$$\boldsymbol{A}^m = \begin{bmatrix} 2m+1 & -4m \\ m & 1-2m \end{bmatrix}.$$

   for all $m \in \mathbb{N}$.

   (b) Hence express $\boldsymbol{A}^m$ linearly in the form

$$\boldsymbol{A}^m = \boldsymbol{I} + m\boldsymbol{B}$$

   for some matrix $\boldsymbol{B}$.

Algebra

Definition 2.
Let $(G, *_G), (H, *_H)$ be groups. A **homomorphism** ϕ is a function $\phi : G \to H$ such that

$$\phi(g_1 *_G g_2) = \phi(g_1) *_H \phi(g_2).$$

If ϕ is also a bijection from G to H, then we say that ϕ is an **isomorphism**.

9. For each of these functions, determine whether or not they are (a) **homorphisms** (b) **isomorphisms**.

   (a) The map $\phi : (\mathbb{Z}_4, +) \to (\mathbb{Z}_8, +)$ with

$$\phi(x) = 2x.$$

   (b) The map $\phi : (\mathbb{Z}_4, +) \to (\mathbb{Z}_8, +)$ with

$$\phi(x) = 2x + 1.$$

   (c) Let R be the set of rotations of the square, with $R = \{\rho_0, \rho_1, \rho_2, \rho_3\}$. Furthermore let

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

   and define another matrix group S where

$$S = \{A^n : n \in \{0, 1, 2, 3\}\}.$$

   Let $\phi : R \to S$ with

$$\phi(\rho_k) = \boldsymbol{A}^k.$$

Definition 3.
The **kernel** of a homomorphism $\phi : G \to H$ is the set

$$\ker \phi = \{g \in G : \phi(g) = \mathrm{id}_H\}.$$

10. Let $\phi : G \to H$ be a homomorphism. Using the definition of the kernel above,

   (a) Prove that $\ker \phi$ is a subgroup of G.

   (b) Prove that ϕ is injective if and only if $\ker \phi = \mathrm{id}_G$.